ORIGIN OF CLASSICAL ELECTROMAGNETIC-MASS DISCREPANCY

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ABSTRACT

The conventional electrodynamic equations are consistent with the usual problem of energy and momentum flow through a boundary. For the problem of integrating the energy and momentum over a moving volume of field, where the anomalous factor of 4/3 frequently arises, the conventional expressions are incorrect. The difference between these two types of problems is closely related to the difference between internal energy and enthalpy in thermodynamics.

INTRODUCTION

Electrodynamics is an extensive body of knowledge that has obvious and substantial utility. There are, however, some problems that have never been adequately resolved - such as the electromagnetic mass of an electron. This problem was studied prior to relativity, but the anomalous nature of the answer was not appreciated until the mass-energy equivalence of relativity was incorporated. When the electromagnetic momentum of an electron is expressed as a product of the translational velocity and the equivalent mass of the electric-field energy, a factor of 4/3 is also obtained. This factor of 4/3 is the essence of the electromagnetic-mass problem of an electron, and a wide variety of explanations have been given for the presence of this factor. Some of these explanations are currently included in a number of widely used textbooks.[1-4]

Although the electromagnetic mass of an electron has received the most attention in recent years, there are actually a number of related problems. The parallel-plate capacitor is a classic problem that was investigated as part of the search for an ether. The investigation of Trouton and Noble[5] was based on the variation of electromagnetic energy with the angle between the field direction and the velocity through the ether. Conservation of energy indicated that a torque was required to supply the energy change with angle. Trouton and Noble looked for this torque, but found nothing beyond residual electrostatic effects. Although the problem was formulated in terms of electromagnetic energy

by Trouton and Noble, it can also be formulated in terms of electromagnetic momentum. This latter formulation was used by Lorentz, who, in relativistic terms, showed that the electromagnetic momentum can be anywhere from zero to twice the product of velocity and equivalent mass of the electric-field energy, again depending on the angle of the electric field with the velocity.[6] In the present framework of relativity, of course, the velocity would be relative to an observer instead of an ether.

The anomalous factor of 4/3 is usually given in connection with the electromagnetic mass of an electron, but it actually applies to any radially symmetric electric field and is closely related to the momentum variation obtained for the rotation of a parallel-plate capacitor. Some of the explanations that have been given for the factor being 4/3 instead of unity in the electromagnetic momentum of an electron are: that the field near a single electronic charge obeys different laws than the macroscopic field volumes normally considered in classical electrodynamics, that there is a non-electromagnetic mass in an electron as well as an electromagnetic mass, and that the difference is accounted for by the radiation reaction. Such explanations can be countered by considering radially symmetric configurations other than the field of an electron. The explanation of unknown laws for the field near a single electronic charge can be countered by assuming a field produced by a large number of charged particles, so the field can be of the macroscopic size that is clearly in the domain of classical

electrodynamics. The possible explanation of the electron having non-electromagnetic mass can be countered by using a charged body with a very large potential, so that the equivalent mass of the electric-field energy is significant compared to the rest mass of the charged particles that produce the field. The radiation reaction explanation can be countered by using a spherical-capacitor configuration with the electric field confined to the volume between the two electrodes. Such a configuration will have negligible radiation at wavelengths long compared to the spacing between the two electrodes, and these wavelengths can be made the important part of the radiation spectrum by using long times for acceleration changes. Because these explanations can be shown unsuitable for the general problem of radially symmetric electric fields, consideration should be given to finding the fault in the basic concepts of electromagnetic momentum and energy - instead of just the application of these concepts.

The approach used herein for the analysis of electromagnetic problems, including the one of momentum, is to construct a physical model for electromagnetic fields. The central idea of this approach is simply to take the conventional concepts and analytical methods used for physical objects and - with a minimum of modification - applying them to fields. In short, treating a field as a physical object. This approach may seem uninspired, but it results in substantial departures from conventional electrodynamic theory. The justification for this approach must be the same as the justification for any new approach - the utility

of the results obtained. The approach used herein should be justified if a class of problems that previously gave anomalous results could thereby be explained. The greater understanding that would presumably accompany such an explanation should also result in a sounder foundation for future work.

By limiting the problems examined in this paper to a single field at a time, moving without acceleration at a velocity small compared to that of light, the essential features of the physical-model approach are displayed with a minimum of theoretical distraction. SI units (rationalized mks system) are used throughout this paper.

CONVENTIONAL ELECTROMAGNETIC APPROACH

In order to compare electromagnetic field mass as indicated by the two theories, it is necessary to recapitulate some electromagnetic theory. Analyzing static fields with physical parameters such as energy density and stress has been found convenient and effective. For an electric field,

$$\rho = \tfrac{1}{2}\epsilon_0 E^2, \tag{1}$$

$$\sigma_{\perp} = \tfrac{1}{2}\epsilon_0 E^2, \tag{2}$$

$$\sigma_{\parallel} = -\tfrac{1}{2}\epsilon_0 E^2. \tag{3}$$

These parameters are the field energy density, the Maxwell stress normal to the electric-field direction, and the Maxwell stress parallel to the electric-field direction. Compression is assumed to be a positive stress, so that tension is a negative quantity. Any three-dimensional electromagnetic field can be resolved into three mutually orthogonal

components, with each component parallel to one of the three orthogonal spatial coordinates. Consideration of field strengths and stresses for only the directions parallel and normal to the field direction can therefore give results of general utility.

The mass-energy equivalence of relativity indicates a field at rest has an equivalent mass of

$$m_\ominus = \mathcal{E}/c^2 ,\qquad (4)$$

where the field energy $\mathcal{E}$ is obtained by integrating the field energy density ρ over the volume of the field τ.

$$\mathcal{E} = \int \rho \; d\tau \qquad (5)$$

The complicated nature of electromagnetic mass becomes evident when moving fields are considered. The energy flow is obtained by integrating the Poynting vector

$$\bar{S} = \bar{E} \times \bar{H} \qquad (6)$$

over the closed surface through which field energy is flowing. The related equation for electromagnetic-momentum density is

$$\bar{g} = \bar{E} \times \bar{H}/c^2 ,\qquad (7)$$

so that the momentum $\bar{p}$ of a field is obtained by integrating $\bar{g}$ over the volume of the field.

$$\bar{p} = \int \bar{g} \; d\tau \qquad (8)$$

Although the preceding two equations are widely used and accepted as definitions of electromagnetic momentum, there are some mathematical ambiguities associated with their use.[7] The momentum problems mentioned in the Introduction, then, might be resolved by a change in these

equations. The approach used herein, though, will only consider the effects of using a physical model.

The fields of both a radially symmetric configuration and a parallel-plate capacitor will be considered in the electromagnetic-momentum calculations. To make the physical implications more evident, however, the simpler model of the parallel-plate capacitor will be considered first. Assume the capacitor consists of two square plates of length 1 on each side, with the spacing s the distance between the parallel plates. The fringing and other external-field energy can be made small compared to the field energy between the plates by assuming s≪1.

If this capacitor moves relative to the observer with a velocity v_o parallel to the electric-field direction, as indicated in figure 1, the
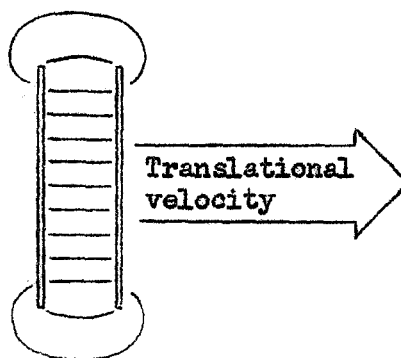

Figure 1. - Velocity parallel to the electric-field direction.

Lorentz field transformation will result in no magnetic field. The momentum density will therefore be zero, and the total momentum of the uniform-field region between the plates is

$$P_{\parallel} = 0 \qquad (9)$$

for the parallel-field orientation.

If the capacitor now moves normal to the electric-field direction, as indicated in figure 2, the Lorentz transformation results in a
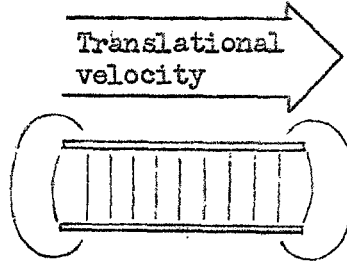

Figure 2. - Velocity normal to the electric-field direction.

magnetic field. Assuming that the velocity v_0 is small compared to the velocity of light c so that the results are applicable to the classical regime, the Lorentz transformation gives the magnetic field as

$$H = \epsilon_0 E v_0. \tag{10}$$

The electromagnetic momentum for the uniform-field volume τ between the parallel plates can be evaluated from equations (7), (8), and (10) as

$$p_\perp = \epsilon_0 E^2 v_0 \tau / c^2. \tag{11}$$

This result obviously differs from equation (9). The nature of this difference can be made clearer by expressing the preceding momentum in terms of the field-energy equivalent mass. For $v_0 \ll c$, the energy in the electric field of this capacitor can be obtained from equations (1) and (5).

$$\mathcal{E} = \tfrac{1}{2}\epsilon_0 E^2 \tau \tag{12}$$

The equivalent mass of this energy is

$$m_e = \tfrac{1}{2}\epsilon_o E^2 \tau / c^2 . \tag{13}$$

If equation (11) is written in terms of this equivalent mass,

$$p_\perp = 2m_e v_o . \tag{14}$$

Equations (9) and (14) show the momentum variation with orientation that was mentioned in the Introduction. As was also indicated in the Introduction, these results have been in existence for many years.

According to the mass-energy equivalence of relativity, a quantity of field energy would be expected to have a momentum $m_e v_o$. Because neither equation (9) nor (14) agrees with this expected value, two interpretations of the results appear possible. Either the mass-energy equivalence does not apply to electromagnetic field energy in the same way that it does to other energies, or there is something wrong with the conventional formulation of electromagnetic momentum. The author's contention is the latter.

The momentum results obtained for a parallel-plate capacitor can easily be applied to a radially symmetric field configuration. Any electric-field energy can be analyzed as three mutually orthogonal components.

$$\tfrac{1}{2}\epsilon_o E^2 = \tfrac{1}{2}\epsilon_o E_x^2 + \tfrac{1}{2}\epsilon_o E_y^2 + \tfrac{1}{2}\epsilon_o E_z^2 \tag{15}$$

Integrated over the entire field volume, the energy of a radially symmetric field can be treated as if it were divided equally between the three components. The velocity v_o can be assumed parallel to one of these components, so that one-third of the field energy can clearly be associated with a component parallel to motion and two-thirds associated

with components normal to motion. Thus one-third of the field energy will, in accord with equation (9), have no momentum. The remaining Two-thirds will, in accord with equation (14), have twice the momentum expected from the equivalent mass of that field energy. The momentum of a radially symmetric configuration, in terms of the equivalent mass of its entire electric-field energy, is therefore

$$p = (4/3)m_e v_o. \tag{16}$$

The factor of $4/3$ instead of unity in the preceding equation is, of course, the basis of the electromagnetic-mass problem for an electron. As indicated in the derivation, though, the result actually applies to any radially symmetric electric-field configuration.

PHYSICAL-MODEL APPROACH

A velocity of definite magnitude and direction can be attributed to every macroscopic physical body by an observer. A physical approach to fields, then, should define a similar velocity for each macroscopic field element. It is true that velocity effects are included in classical electrodynamics by the Lorentz field transformations. As shown in the preceding section, though, this inclusion of velocity effects does not give a field momentum that is consistent with our concepts of momentum for physical bodies.

A field velocity is easy to define for some field configurations, and more difficult for others. For example, the electric-field energy of a charged body extends over a large and perhaps infinite volume, but the bulk of it is localized near the body. Moving a charged body from

one location to another will thus move energy in a similar manner. If the field of a charged body is to be treated as a physical object, it must - with allowance for propagation of acceleration disturbances - move with the charged body. A charged parallel-plate capacitor would be another example of the field motion being clearly associated with the motion of a physical body.

A moving field can be thought of as a moving fluid, with the word "fluid" used in its most general sense. To describe the fluid properties of an electric or magnetic field in conventional terminology, it has no viscosity and it can be converted completely into work. The entropy of a classical electromagnetic field must therefore be zero, and any flow process for a field must be isentropic. One other major departure from most fluids is the non-isotropic nature of field stresses.

The rigorous treatment of energy transfer with a moving fluid has long been a part of thermodynamics, and for this reason certain concepts of thermodynamics will be used as a basis for energy-transfer calculations with a moving field. Treatment of thermodynamic energy-transfer problems can be divided into flow and non-flow categories, depending on whether or not the energy-evaluation boundary is assumed to be stationary relative to the fluid. The essential feature of a flow treatment is the evaluation of the rate at which energy crosses a boundary with a moving fluid, while a non-flow treatment is concerned with the integration of energy over the volume enclosed by the boundary.

Many problems can be approached with either a flow or non-flow
treatment, as long as the treatment is applied in a rigorous manner.
Certain problems, though, are particularly suited for one of the two
treatments and, in this sense, may be called flow or non-flow problems.

A non-flow treatment in electrodynamics would be one in which the
boundary is located so that the field does not move across it. A moving
capacitor falls conveniently in the non-flow category by assuming a
boundary far from the capacitor, so that no significant amount of field
falls outside the boundary. Although motion of a boundary relative to
an observer is not usually considered in thermodynamics, the absence of
flow across that boundary is sufficient to define the treatment as non-
flow. The boundary could thus move with the moving capacitor and the
problem would still be in the non-flow category.

The energy of a fluid in the non-flow thermodynamic treatment is
simply the total internal energy of that fluid in the volume under
consideration. For an electric field with $v_0 \ll c$, the "internal energy"
can be obtained from the energy density of equation (1). The total
energy in the uniform-field volume τ between the electrodes of a
parallel-plate capacitor is thus

$$\mathcal{E} = \tfrac{1}{2}\epsilon_0 E^2 \tau. \tag{17}$$

The equivalent mass of this energy is

$$m_e = \tfrac{1}{2}\epsilon_0 E^2 \tau / c^2. \tag{18}$$

(The kinetic energy of the moving field $m_e v_0^2/2$ would result in an
additional mass of $m_e v_0^2/2c^2$. The assumption of $v_0 \ll c$, though, makes

these terms negligible compared to the energy and equivalent mass shown in equations (17) and (18).) The momentum that would normally be associated with the motion of this mass at velocity v_0 is $m_e v_0$, or

$$p = \tfrac{1}{2}\epsilon_0 E^2 v_0 \tau / c^2 . \tag{19}$$

Because energy density is not a function of field direction, field direction has no effect on this momentum. Also, although this momentum is the value that would be expected from the mass-energy equivalence, it agrees with neither equation (9) nor (11). As indicated earlier, the author prefers a strict adherence to the mass-energy equivalence when a choice must be made between this equivalence and the usual definition of electromagnetic momentum (equation (8)). Equation (19) is therefore the form preferred by the author for the momentum of the electric-field energy in a parallel-plate capacitor.

A radially symmetic electric-field configuration, such as an electron, also falls conveniently in the non-flow category. It is true that the field of an electron extends for an indefinite distance, but the bulk of the field energy - and thus the field momentum - is confined to a small volume near the electron. The electron itself therefore defines the average location of the field energy of the electron, and a boundary located far from the electron will enclose substantially all of this field energy. Use of the non-flow approach for an electron, with $v_0 \ll c$, gives a total electric-field energy substantially the same as its static-field value. The momentum of an electron would be simply the product of the equivalent mass m_e of this total energy and the velocity

v_0. The factor 4/3 would therefore not appear when the momentum of the electron is evaluated using non-flow concepts.

Energy transfer in the thermodynamic flow treatment involves not just internal energy, but also flow work. Flow work is the energy required to push the fluid across the boundary, as indicated in figure 3.
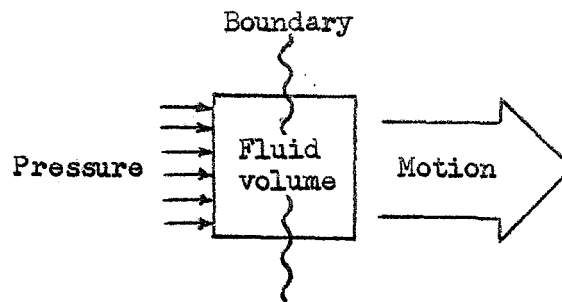
Flow work = (Pressure) X (Fluid volume)

Figure 3. - Flow work

The pressure is isotropic in the fluids normally considered in thermodynamics, but only the component of pressure in the direction of fluid motion contributes to the flow work. Inasmuch as the internal energy of a field is typically given per unit volume, the flow work should also be evaluated in this manner. The flow work per unit volume of a field is simply the field stress parallel to the field motion involved. The boundary for energy-flow evaluation must be a closed surface to assure inclusion of all flow processes. The boundary in a thermodynamic flow treatment is also usually assumed stationary relative to the observer, which, in general, will be convenient for electromagnetic problems.

The radiation of electromagnetic energy is obviously suited to a flow treatment. The boundary can be at the radiation source, at an absorber, or at any surface of energy-flow evaluation between the two. Electromagnetic radiation, though, travels at the velocity of light, which is beyond the scope of this paper.

For a flow problem in the classical regime, an electric field can be assumed to be moving across a boundary at a velocity small compared to that of light. For example, the slow addition of electrical charge to a charged body would presumably involve an electric field moving at low velocity. The total energy flow across a unit area of boundary normal to the field velocity v_0 is, from thermodynamics, the sum of internal-energy and flow-work terms. The energy flow S_f for an electric or magnetic field can thus be defined

$$S_f = (\rho + \sigma)v_0. \tag{20}$$

With $v_0 \ll c$, the kinetic energy term usually given for moving fluids in thermodynamics is negligible compared to the internal energy term. The stress σ is, of course, the stress in the direction of v_0. The quantity $\rho + \sigma$ is analogous to enthalpy, a state function in thermodynamics. The non-isotropic nature of electromagnetic stresses, though, prevents the definition of a similar state function in electromagnetic problems.

The significance of equation (20) can be demonstrated by considering an electric field with various orientations relative to the velocity. As mentioned earlier in this paper, any electric or magnetic field can be resolved into orthogonal components. Consideration of

field orientations parallel and normal to the field velocity will therefore give results of general utility. Assume first an electric field parallel to v_o, and use equations (1) and (3) to substitute for energy density and stress. The energy density and stress cancel to give a net energy flow of

$$(S_f)_\parallel = 0. \tag{21}$$

For a field normal to v_o, equation (2) should be used for the stress. The internal energy and stress add in this case, giving

$$(S_f)_\perp = \epsilon_o E^2 v_o. \tag{22}$$

Both of the preceding results agree with the conventional approach using the Poynting vector of equation (6). In conventional terminology, the electric field is unchanged by a velocity parallel to itself and there is no magnetic field. With no magnetic field, there can be no energy flow. For the transverse orientation, the velocity v_o normal to the electric field gives a magnetic field of $\epsilon_o E v_o$, so that the Poynting vector is $\epsilon_o E^2 v_o$.

The energy flow S_f can thus be identified with the Poynting vector S. That is, the conventional electrodynamic formulation of energy flow agrees with the thermodynamic flow treatment. It is significant that the admonition frequently given in electrodynamics - to make sure that the boundary for energy-flow evaluation is a closed surface - is the same one given for the flow treatment in thermodynamics.

Although the energy density of the electric field is defined by equation (1), a similar quantity ρ_f can be defined using the preceding

energy flow equations.

$$\rho_f = S_f/v_o \tag{23}$$

Dividing this "energy density" by c^2 to obtain a "mass density", and

then multiplying by v_o gives a momentum density g_f.

$$g_f = S_f/c^2 \tag{24}$$

With the substitution of the energy flows from equations (21) and (22),

$$(g_f)_{\parallel} = 0, \tag{25}$$

$$(g_f)_{\perp} = \epsilon_o E^2 v_o/c^2. \tag{26}$$

These results agree with the conventional momentum density of

equation 7. The momentum density g_f can thus be identified with the

electromagnetic momentum density g.

The statement was made earlier in this paper that many problems can

be approached with either a flow or non-flow treatment. For example, a

moving parallel-plate capacitor is well suited for a non-flow treatment,

but it can also be analyzed with a flow treatment. If the parallel-

plate capacitor is assumed to be moving with the field parallel to the

velocity (fig. 1), the internal energy and stress for the electric field

cancel in equation (20). The net energy flow from the electric field is

therefore zero, as shown in equation (21). The boundary for energy-flow

evaluation must be a closed surface, which means that all flow energies

crossing a boundary must be included. A parallel-plate capacitor must

have insulating supports to resist the tension parallel to the field, as

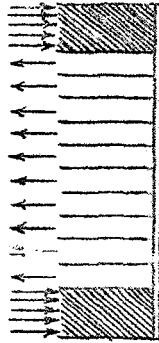indicated in figure 4. For equilibrium between the two forces, the

Figure 4. - Opposition of field and support forces.

tension force of the field must be balanced by the compression force of
the supports. The flow work of a volume moving across a boundary can
also be expressed as the product of a force (the stress times the area)
and the distance over which the force acts. The equality of tension and
compression forces thus results in the support flow work canceling the
field flow work, leaving only the internal energy of the non-flow
treatment.

For the transverse field orientation (fig. 2), the compression
force of the field is balanced by the tension force of the capacitor
plates, as indicated in figure 5. The capacitor-plate flow work thus

Figure 5. - Opposition of field and capacitor-plate forces.

cancels the field flow work, again leaving only the internal energy of the electric field.

In a similar manner for a charged spherical body, the tension force in the body will yield a flow work that will cancel the flow work of the surrounding field. The net energy will thus again equal the internal energy of the non-flow treatment.

In case the reader wonders about the distortion energy stored in these mechanical parts, this distortion energy is proportional to the strain (strain $= \Delta x/x$) of these parts in the direction of the applied force. The mechanical parts can be assumed to be very stiff, so that they are only microscopically strained by the electric field forces. The distortion energy is then negligible compared to other energies.

CONCLUDING REMARKS

The calculations of both the conventional and the physical-model approaches in this paper have used an electric field as a starting point. The similarity of electric and magnetic equations, though, is such that similar results would have been obtained if a magnetic field had been used as a starting point. A long solenoid of uniform cross section could have been used in place of the parallel-plate capacitor. The classical expression for momentum density would again indicate a variation of field momentum with field orientation. The use of the physical-model approach would result in the same use of internal energy for the non-flow approach, and the same equality to the results of classical electrodynamics for the flow treatment. The advantages found

for the physical-model approach with electric fields will therefore also be found with magnetic fields.

For the regime considered in this paper of a single electric or (by implication) magnetic field moving without acceleration at a velocity small compared to that of light, the physical-model approach resolved the energy and momentum discrepancies of a number of problems. The conventional energy-flow and momentum-density expressions were found to correspond to the flow treatment of the physical-model approach. The problems in which energy and momentum discrepancies were obtained were most conveniently approached with a non-flow treatment. The difference between these two types of treatments is closely related to the difference between internal energy and enthalpy in thermodyanmics. The lack of the flow-work concept in classical electrodynamics might therefore be considered the source of the electromagnetic-mass problem.

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